A continuous-time model of the term structure of interest rates with fiscal-monetary policy interactions

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Abstract

We study the term structure implications of the fiscal theory of price level determination. We introduce the intertemporal budget constraint of the government in a general equilibrium model in continuous time. Fiscal policy is set according to a simple rule whereby taxes react proportionally to real debt. We show how to solve for the prices of real and nominal zero coupon bonds.

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1 Introduction

The theory of price level determination advocated by Leeper (1991), Sims (1994), Woodford (1996) and Cochrane (1998) has brought to the attention of macroeconomists the role of interactions between fiscal and monetary policy. In a nutshell, the idea is that the price level is determined by the degree of solvency of the government. If the expected primary surplus is not sufficient to comply with the intertemporal budget constraint of the government, then part of the public debt should be inflated away if it is default-free.

Although the fiscal theory of price level determination has generated a substantial debate on the capability of fiscal and monetary policy to affect the price level, study has considered its potential implications for asset prices. This considerations holds both for the finance and macroeconomics literature. For instance, the continuous-time model of the term structure of interest proposed by Buraschi (2005) includes lump-sum taxes, but disregards the implications of the government's budget constraint. Dai and Philippon (2005) estimates a no-arbitrage ane term structure model with fiscal variables on U.S. data. They find significant responses of the term structure of interest rates to the deficit-GDP ratio. The macroeconomic restrictions they impose to identify the structural responses are fairly different from those implied by the fiscal theory of the price level (see Sala, 2004).

The available finance models the term structure of interest rates consider an explicit role for only two crucial factors, output growth and monetary policy, which is typically expressed as a diffusion process for the growth of money supply. In this paper, we consider a general-equilibrium model with money where the ow budget constraint of the government plays an active role. This provides a link between monetary and fiscal policy because lump-sum taxes are adjusted as a function of real debt. We solve the structural model, and derive the law of motion for the nominal and real interest rates. We also study how the term structure responds to the fiscal parameters.

This paper is organized as follows. The first two sections introduce the reader to the framework employed to develop the analysis, together with a brief discussion on the fiscal and monetary policy rules adopted. Section 4 and 5, respectively, discuss the optimization process form the representative investor's side and the characterization of the equilibrium. Section 6 outlines the continuous time limit of the equilibrium relationships in discrete time presented in the previous sections. In section 7, we consider a specialized economy with a more realistic set of assumptions for the model. In section 8 we present the solution for the real spot rate. This is extended in section 9 for the pricing of the entire real term structure. The nominal and real term structure for zero coupon bonds is derived in section 9. Since the solution does not admit a closed form, we use numerical simulations in section 10 to generate some qualitative results on the shape of the term structure. Section 11 reports some concluding remarks.

2 The model economy

We study an economy populated by a representative agent that maximizes over the composition of her portfolio along the lines of the traditional literature on consumption and asset pricing. We model the economy at discrete time intervals of length Δt. The representative agent chooses its portfolio holdings by maximizing the following utility function

$$\sum_{t=0}^{\infty} e^{-\beta t} E_0 \left\{ u\left(C_t, \frac{M_t}{P_t} \right) \right\} \Delta t \tag{1}$$

where β is the discount factor. In equation (1), C_t indicates the level of consumption over the interval $[t, t + \ \Delta t]$, M_t is the nominal money stock providing utility to the representative agent over the interval of length $[t - \Delta, t]$, and P_t is the price of the consumption good. Real money balances M_t/P_t enter the utility function of the household. The utility function is twice continuously differentiable and concave in both consumption and real balances

$$u_c > 0,\ u_m > 0,\ u_{cc} < 0,\ u_{mm} < 0,\ u_{cm} < 0,\ u_{cc}u_{mm} - (u_{cm})^2 > 0 \tag{2}$$

where the subscript to u indicates the partial derivative. In what follows, we make the following functional assumption on the utility function

$$u\left(C_t, \frac{M_t}{P_t}\right) = \phi \log C_t + (1-\phi)\log\left(\frac{M_t}{P_t}\right) \tag{3}$$

This type of utility function is used in Stulz (1986). In equation (3), the preference parameter ϕ must be chosen so that the nominal and real spot rates determined under the assumption of absence of arbitrage opportunities are also equilibrium values (see Corollary 1 in the Appendix).

As a working hypothesis to derive the first order conditions, we consider a model of pure endowment economy where output growth evolves as

$$\frac{\Delta Y_t}{Y_t} = \frac{Y_{t+\Delta t} - Y_t}{Y_t} = \mu_{Y,t}\Delta t + \sigma_{Y,t}\Omega_{Y,t}\sqrt{\Delta t}. \tag{4}$$

The terms $\mu_{Y,t}$ and $\sigma_{Y,t}$ are, respectively, the conditional expected value and the standard deviation of output per unit of time and $\{\Omega_{Y,t} t = 0, \Delta t, \ldots\}$ is a standard normal process.[1]

3 Fiscal and monetary policy

The main point of this paper is to examine the impact of the interaction between monetary and fiscal policy on the the term structure of interest rates. We think of 'interactions' in the sense captured by the "fiscal theory of the price level" of Leeper (1991), Sims (1994), Woodford (1996), and recently extended by Cochrane (1998, 1999). This approach states that a tight fiscal policy is a strictly necessary complement to ensure price stability.

[1] A more realistic law of motion for output is introduced in section 7.

We define the money supply aggregate (in nominal terms) as

$$M_t^s = H_t + F_t. \tag{5}$$

In equation (5) we observe that the total money supply is determined by two components. H_t is the so called 'high powered money' (or monetary base). F_t represents the amount of money needed by the government to budget its balance. Basically, F_t is an additional financing source for the government apart from taxes and debt[2].

We assume that H_t and F_t follow the processes described by

$$\frac{\Delta H_t}{H_t} = \frac{H_{t+\Delta t} - H_t}{H_t} = \mu_{H,t}\Delta t \tag{6}$$

$$\frac{\Delta F_t}{F_t} = \frac{F_{t+\Delta t} - F_t}{F_t} = \mu_{F,t}\Delta t + \sigma_{F,t}\Omega_{F,t}\sqrt{\Delta t} \tag{7}$$

where $\mu_{H,t}$ and $\mu_{F,t}$ are, respectively, the mean of the stochastic process of the monetary base and of the financing to public debt. In (6), the stochastic process for H_t does not have a standard error term, implying that the monetary base possesses only a deterministic component. The process leading F_t, instead, has a standard deviation term $\sigma_{F,t}$, where $\{\Omega_{F,t} t = 0, \Delta t, \ldots\}$ are standard normal random variables.

From (5), (6) and (7), we can write the stochastic process for the total money supply M^s

$$\frac{\Delta M_t^s}{M_t^s} = \frac{M_{t+\Delta t}^s - M_t^s}{M_t^s} = \mu_{M,t}\Delta t + \sigma_{M,t}\Omega_{M,t}\sqrt{\Delta t} \tag{8}$$

$$\mu_{M,t} = \mu_{H,t} + \mu_{F,t} \tag{9}$$

$$\sigma_{M,t}\Omega_{M,t} = \sigma_{F,t}\Omega_{F,t}. \tag{10}$$

At a first glance, these expressions stress that the central bank is assumed to target money growth.

The subsequent building block of the model assigns a proper macroeconomic role to the government. The innovation introduced in this paper with respect to the

[2] F_t can be thought of as the demand for money balances expressed by the government.

existing literature consists in the key role for the government budget constraint

$$\Delta D_{t+\Delta t} + \Delta F_{t+\Delta t} = \Delta i_{t+\Delta t} D_t - \Delta T_{t+\Delta t} \tag{11}$$

where D_t indicates the stock of public debt, and $\Delta i_{t+\Delta t}$ is the stochastic process of the nominal spot interest rate, whose endogeous law of motion will be computed later. Moreover, $\Delta T_{t+\Delta t}$ is the stochastic process for taxes. We assume that the government does not face any form of public spending. Recall that $\Delta D_{t+\Delta t} = D_{t+\Delta t} - D_t$, $\Delta F_{t+\Delta t} = F_{t+\Delta t} - F_t$. Basically, the government can use taxes, money and debt to finance its budget.

Following the fiscal theory of price level, we assume that the government sets taxes according to the simple rule rule

$$\Delta T_{t+\Delta t} = \phi_1 D_t \Delta t + \phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t} \tag{12}$$

According to (12), the government sets as a function of the outstanding amount of public debt. This means that if the stock of debt issued rises, taxes must change accordingly with a marginal elasticity equal to ϕ_1. A bound on ϕ_1 can be established by following Sims (1994) by setting ϕ_1 at a value less than or equal to the discount factor β. To close the model, the process for the nominal spot interest rate follows

$$\Delta i_{t+\Delta t} = \mu_i \Delta t + \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t}, \tag{13}$$

for values of the mean and the standard deviations to be determined later. By plugging (13) and (12) into (11), we can recover the flow budget constraint of the public sector

$$\Delta D_{t+\Delta t} + \Delta F_{t+\Delta t} = (\mu_i - \phi_1) D_t \Delta t + D_t \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t} - \phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t}. \tag{14}$$

In order to obtain a semi-closed form solution, we assume that the quantity of newly-issued public debt follows a deterministic process with mean μ_D

$$\frac{\Delta D_t}{D_t} = \frac{D_{t+\Delta t} - D_t}{D_t} = \mu_{D,t} \Delta t. \tag{15}$$

Thus, the flow budget constraint becomes

$$\begin{aligned} \mu_{D,t} \Delta t + \mu_{F,t} \Delta t + \sigma_{F,t} \Omega_{F,t} \sqrt{\Delta t} = \left(\mu_i - \phi_1\right) D_t \Delta t \\ + D_t \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t} + -\phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t}. \end{aligned} \tag{16}$$

To get intuition of how these relation work, concentrate on their deterministic part. Assume that the government aims to maintain a constant ratio of nominal bond to governmental money, i.e., $\psi = D/F$. Therefore, by applying Ito's Lemma to the definition of ψ, we can write the relationship between the mean of the public debt and money

$$\mu_D = \mu_F - \sigma_F^2. \tag{17}$$

From the equality between the deterministic and the stochastic terms of F_t,

$$\psi \mu_D + \mu_F = \left(\mu_i - \phi_1\right) \psi \tag{18}$$

$$\sigma_F = \psi \left(\sigma_i - \phi_1 \sigma_T\right). \tag{19}$$

Finally, using the definition of μ_D into (18), we get the semi-closed solution for the mean of the stochastic process of the governmental money

$$\mu_F = \frac{\left(\mu_i - \phi_1 + \sigma_F^2\right) \psi}{1 + \psi}. \tag{20}$$

Therefore, (19) and (20) represent the full equilibrium relationship in the economy. By using (20), it is clear that the mean of the stochastic process leading money is

$$\mu_M = \mu_H + \frac{\left(\mu_i - \phi_1 + \sigma_F^2\right) \psi}{1 + \psi}. \tag{21}$$

4 The optimal choice problem

The representative agent's budget constraint is

$$M_t + \left(P_{z,t} + P_t^C y_t \Delta t\right) z_t + P_t^C a_{1,t} + a_{2,t} + \sum_{i=3}^{N} P_{i,t} a_{i,t} = P_t^C C_t \Delta t + M_{t+\Delta t}$$
$$+ P_{z,t} z_{t+\Delta t} + P_t^C \frac{a_{1,t+\Delta t}}{1 + r_t \Delta t} + \frac{a_{2,t+\Delta t}}{1 + i_t \Delta t} + \sum_{i=3}^{N} P_{i,t} a_{i,t+\Delta t} \quad (22)$$

The investor can choose among one real and one nominal bond (both risk free), and $N-2$ equities. Each bond is issued at time t and has maturity at time $t + \Delta t$. The return on bond are i_t for the nominal bond, and r_t for the real bond. $P_{i,t}$ is the price (inclusive of dividends) of asset i at time t. The representative agent demands M_t, for cash, C_t for consumption and x_t for equity holdings. $a_{1,t}, a_{2,t}, \ldots a_{N,t}$ represent the unit of financial asset held from $(t - \Delta t)$ to t.[3]

The choice problem of the representative investor consists in the maximization of the utility function (3) subject to the budget constraint (22). The first order conditions for C_t, $a_{1,t}$, $a_{2,t}$, M_t and $a_{i,t}$ are, respectively,

$$u_c\left(C_t, m_t\right) = \lambda_t P_t^C \quad (23)$$

$$E_t\left[e^{-\beta \Delta t} \lambda_{t+\Delta t} P_{t+\Delta t}^C \left(1 + r_t \Delta t\right)\right] = \lambda_t P_t^C \quad (24)$$

$$E_t\left[e^{-\beta \Delta t} \lambda_{t+\Delta t} \left(1 + i_t \Delta t\right)\right] = \lambda_t \quad (25)$$

$$E_t\left[e^{-\beta \Delta t} \lambda_{t+\Delta t} + u_m\left(C_{t+\Delta t}, m_{t+\Delta t}\right) \frac{1}{P_{t+\Delta t}^C}\right] = \lambda_t \quad (26)$$

$$E_t\left[e^{-\beta \Delta t} \lambda_{t+\Delta t} P_{i,t+\Delta t}\right] = \lambda_t P_{i,t} \quad (27)$$

[3]This setup above described is similar to that of Baksi and Chen (1996), who use this model to study the impact of monetary policy and inflation on financial asset.

5 Definition of equilibrium

We assume that the economy is populated by identical agents. In a representative agent economy, optimal consumption, money demand and portfolio holdings must adjust in order that the following equilibrium conditions are verified in general equilibrium

$$C_t = Y_t \tag{28}$$

$$M_t \equiv M_t^s = M_t^d \tag{29}$$

$$z_t = 1 \tag{30}$$

$$a_{i,t} = 0 \quad \forall\, i = 1, \ldots N \tag{31}$$

In a pure endowment economy, (28) states that the total amount of consumption must equal the total output endowment. The equality between money demand and supply is stated in equation (29), while equation (30) states that each agent's demand for equity shares must equal the supply. In the same way, each agent's demand for financial assets should equal supply which is zero, as showed in (31).

Using equations (28)-(31) and the first order conditions (23)-(27), we obtain

$$u_c\left(Y_t, m_t\right) = e^{-\beta\Delta t} E_t\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\left(1 + r_t \Delta t\right)\right] \tag{32}$$

$$\frac{u_c\left(Y_t, m_t\right)}{P_t^C} = e^{-\beta\Delta t} E_t\left[\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{P_{t+\Delta t}^C}\left(1 + i_t \Delta t\right)\right] \tag{33}$$

$$u_c\left(Y_t, m_t\right) = e^{-\beta\Delta t} E_t\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\frac{p_{i,t+\Delta t}}{p_{i,t}}\right] \tag{34}$$

$$u_c\left(Y_t, m_t\right) = e^{-\beta\Delta t} E_t\left\{\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right) + u_m\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\right]\frac{P_t^C}{P_{t+\Delta t}^C}\right\} \tag{35}$$

where $m_t \equiv \frac{M_t}{P_t^C}$ is the real cash balances and $p_{i,t} \equiv \frac{P_t^i}{P_t^C}$ is the real price (in terms of the consumption goods) of asset i at time t.

Equations (32)-(35) are Euler conditions derived from the utility maximization problem of the representative investor. Equations (32) and (35) state that the representative investor must be indifferent between investing an amount of money equal to P_t^C in a real risk-free bond and holding the same amount in cash. This arbitrage condition holds also for nominal bonds (see equations (33) and (35)). Finally, equations (34) and (35) describe the relation of indifference between investing one more amount of cash of size P_t^C in asset i and holding the same amount in a pure cash. Equations (32)-(35) establish the demand for real money, while (33) and (35) yield the demand for money in nominal terms. Equation (35) states that, in equilibrium, the agent is indifferent between holding P_t^C amount of cash and consuming one extra unit of the good, because both actions produce the same marginal utility. The link between the price level and monetary policy is established by equation (35). Monetary policy and the asset market are tied together through equations (35) and (34), which establish the consistency between the money supply and asset markets. Finally, the interdependence between monetary policy and the goods market is described by equations (32) and (35).

Equation (34) must to hold also for the equity Z_t when we replace $\frac{p_{i,t+\Delta t}}{p_{i,t}}$ with $\frac{p_{z,t+\Delta t}+Y_{t+\Delta t}}{p_{z,t}}$, where $p_{z,t} = \frac{P_{z,t}}{P_t^C}$ is the real price of the equity share. Additional sufficient conditions for the existence of an interior optimum are the two transversality conditions

$$\lim_{T \longrightarrow \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c\left(Y_T, m_T\right)}{u_c\left(Y_t, m_t\right)} p_{i,t} \right\} = 0 \tag{36}$$

$$\lim_{T \longrightarrow \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c\left(Y_T, m_T\right)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_t^C} \right\} = 0 \tag{37}$$

The equality (36) rules out bubbles in the price level of any risky asset. Condition (37), instead, prevents bubbles in the price level from taking place. The intuition behind the two TVCs is if (36) is violated the agent is willing to sacrifice actual consumption in favor of future consumption derived from proceeds from investment in risky assets without bound. Under condition (37), the agent accepts a reduction in consumption today in exchange for a larger amount of money in the future without bound.

To conclude the characterization of the equilibrium relations, we need to define the stochastic process for real asset prices. We follow Baksi and Chen (1996), Merton (1971) and Grossman and Grossmann and Shiller (1982) by assuming

$$\frac{\Delta p_{i,t}}{p_{i,t}} = \mu_{i,t}^{e}\Delta t + \sigma_{i,t}^{e}\Omega_{i,t}^{e}\sqrt{\Delta t} \tag{38}$$

where $\mu_{i,t}^{e}$ and $\sigma_{i,t}^{e}$ are, respectively, the conditional expected value and the standard deviation of real return on asset i per unit of time. Finally, the process $\left\{\Omega_{i,t}^{e} t = 0, \Delta t, \ldots\right\}$ is a standard normal.

6 The equilibrium in the continuous time limit

In this section we characterize the equilibrium for the continuous time limit. These results are independent from the assumptions made on the role of fiscal and monetary policies in the determination of the equilibrium. For this reason, the results from this section are similar to Baksi and Chen (1996) Balduzzi (1998).

Proposition 1 *The equilibrium risk premiums for any risky asset over the real spot interest rate is*

$$\mu_{i,t}^{e} - r_t = -\frac{C_t u_{cc}}{u_c} cov_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dY_t}{Y_t}\right) - \frac{m_t u_{cm}}{u_c} cov_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dm_t}{m_t}\right). \tag{39}$$

Proof 1 *Subtract equation (32) from (34), manipulating the resulting expression and using the definition of the stochastic process for* $p_{i,t}$ *(38) , we get*

$$e^{-\beta\Delta t} E_t\left\{\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)}\left[\left(\mu_{i,t}^{e} - r_t\right)\Delta t + \sigma_{i,t}^{e}\Omega_{i,t}^{e}\sqrt{\Delta t}\right]\right\} = 0. \tag{40}$$

Thus, by taking a Taylor expansion of the equation (40) around steady state, we get

$$\begin{gathered} e^{-\beta\Delta t} E_t\left\{\left[\left(\mu_{i,t}^{e} - r_t\right)\Delta t + \sigma_{i,t}^{e}\Omega_{i,t}^{e}\sqrt{\Delta t}\right]\right. \\ \left.\times\left[1 + \frac{u_{cc}\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)Y_t}{u_c\left(Y_t, m_t\right)}\frac{\Delta Y_t}{Y_t} + \frac{u_{cm}\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)m_t}{u_c\left(Y_t, m_t\right)}\frac{\Delta m_t}{m_t}\right]\right\}\frac{1}{\Delta t} = 0. \end{gathered}$$

By letting $\Delta t \to 0$ and applying Ito's multiplication rule, we obtain (39). ■

Proposition 2 *The real price for the equity share is*

$$P_{z,t} = E_t \int_t^\infty e^{-\beta(s-t)} \frac{u_c\left(Y_s, m_s\right)}{u_c\left(Y_t, m_t\right)} Y_s ds \tag{41}$$

Proof 2 *Recalling that*

$$\frac{P_{z,t+\Delta t} + Y_{t+\Delta t}\Delta t}{P_{z,t}} = \frac{\Delta p_{i,t+\Delta t}}{p_{i,t}}, \tag{42}$$

use equation (34) to obtain

$$P_{z,t} = E_t \left\{ e^{-\beta\Delta t} \frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \left(Y_{t+\Delta t}\Delta t + P_{x,t+\Delta t}\right) \right\}. \tag{43}$$

After iterating forward, the result is

$$P_{z,t} = E_t \sum_{j=1}^{\infty} e^{-\beta(j\Delta t)} \frac{u_c\left(Y_{t+j\Delta t}, m_{t+j\Delta t}\right)}{u_c\left(Y_t, m_t\right)} Y_{t+j\Delta t}\Delta t. \tag{44}$$

Thus, by taking the limit for $\Delta t \to 0$ in (44) we finally get the result under (41). ■

Proposition 3 *In the continuous time limit equilibrium, the commodity price level is given at time t by*

$$\frac{1}{P_t^C} = E_t \int_t^\infty e^{-\beta(s-t)} \frac{u_m(Y_s, m_s)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_s^C} ds \tag{45}$$

The expected inflation rate is

$$\pi_t \equiv \frac{1}{dt} E_t \left\{ \frac{dP_t^C}{P_t^C} \right\} = \tag{46}$$

$$= i_t - r_t + var_t \left\{ \frac{dP_t^C}{P_t^C} \right\} - \frac{u_{cc}Y_t}{u_c} cov_t \left(\frac{dY_t}{Y_t}, \frac{dP_t^C}{P_t^C} \right) \tag{47}$$

$$- \frac{u_{cm}m_t}{u_c} cov_t \left(\frac{dP_t^C}{P_t^C}, \frac{dm_t}{m_t} \right) \tag{48}$$

Proof 3 *Rewrite the first order condition (35) as follows*

$$\frac{1}{P_t^C} = e^{-\beta\Delta t} E_t \left\{ \left[\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_{t+\Delta t}^C} + \frac{u_m\left(C_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \right] \frac{\Delta t}{P_{t+\Delta t}^C} \right\} \tag{49}$$

then, iterate equation (49) to get

$$\frac{1}{P_t^C} = E_t \left\{ \sum_{j=1}^{\infty} e^{-\beta(j\Delta t)} \frac{u_m\left(C_{t+j\Delta t}, m_{t+j\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \frac{\Delta t}{P_{t+j\Delta t}^C} \right\} \tag{50}$$

Taking the limit of the equation (50) we get the result under (45).

To derive the inflation rate, divide the first order conditions (32) and (33)

$$E_t \left\{ \frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \left(1 + r_t \Delta t\right) \right\} = E_t \left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\left(1 + i_t \Delta t\right) \frac{P_t^C}{P_{t+\Delta t}^C} \right] \tag{51}$$

After taking the Taylor approximation of (51) and re-arranging,

$$\begin{aligned} \left(i_t - r_t\right)\Delta t &= \left[1 + \frac{u_{cc}Y_t}{u_c}\left(\frac{\Delta Y_t}{Y_t}\right) + \frac{u_{cm}}{u_c} m_t \left(\frac{\Delta m_t}{m_t}\right)\right]\left[\frac{\Delta P_t^C}{P_t^C} - \left(\frac{\Delta P_t^C}{P_t^C}\right)^2\right] && (52) \\ &\quad + o\left(\Delta t\right)^{3/2} && (53) \end{aligned}$$

Thus, by taking the limit of equation (53), for $\Delta t \to 0$ *we have*

$$\begin{aligned} i_t - r_t = \frac{1}{dt} E_t \left\{ \frac{dP_t^C}{P_t^C} \right\} - var_t \left\{ \frac{dP_t^C}{P_t^C} \right\} + \frac{u_{cc}Y_t}{u_c} cov_t \left(\frac{dY_t}{Y_t}, \frac{dP_t^C}{P_t^C} \right) + \\ + \frac{u_{cm}m_t}{u_c} cov_t \left(\frac{dP_t^C}{P_t^C}, \frac{dm_t}{m_t} \right) \end{aligned}$$

Thus, by using $\pi_t = \frac{1}{dt} E_t \left\{ \frac{dP_t^C}{P_t^C} \right\}$, *and rearranging we get the equation (48).* ■

7 A specialized economy

In what follows we lay out a specific model used to derive the stochastic processes for the price level and the other variables. We assume that the stochastic process for output is

$$dY_t = (\mu_Y + \eta_Y x_t)\, dt + \sigma_Y \sqrt{x_t} dW_{x,t}. \tag{54}$$

where the process for the technology factor x_t is

$$dx_t = a\,(b - x_t)\, dt + \sigma_x \sqrt{x_t} dW_{x,t}, \tag{55}$$

where $(W_{x,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion, μ_Y, η_Y, σ_Y, a, b, and σ_x are fixed real numbers.

The monetary aggregates H_t and F_t follow the exogenous processes

$$d\ln H_t = \mu_H^* dt + d\ln(q_t) \tag{56}$$

$$d\ln F_t = \overline{\mu}_F dt + d\ln(q_t) \tag{57}$$

where q_t is the detrended money supply process. Each type of money supply has two components, a drift term and a stochastic part. In particular, μ_H^* is assumed to be constant and positive, while $\overline{\mu}_F$ is determined by equation (20). From Baksi and Chen (1996), q_t evolves according to

$$\frac{dq_t}{q_t} = k_q\,(\mu_q - q_t)\, dt + \sigma_q \sqrt{q_t} dW_{i,t}, \quad i = H, F \tag{58}$$

where $(W_{i,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent upon $(W_{x,t})_t$. Therefore, by using the definition of money supply (5), we find that the stochastic process leading money supply is

$$\frac{dM_t}{M_t} = \mu_{M,t} dt + \sigma_q \sqrt{q_t} dW_{M,t}. \tag{59}$$

$(W_{M,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent from $(W_{x,t})_t$ and $(W_{i,t})_t$, and where

$$\mu_M = \mu_M^* + 2k_q\,(\mu_q - q_t)\,, \tag{60}$$

with $\mu_M^* = \mu_H^* + \overline{\mu}_F$, and $d\Omega_{M,t} = d\Omega_{H,t} + d\Omega_{F,t}$.

The set of assumptions presented earlier allows us to compute the equilibrium price level of the commodity and the inflation process.

Theorem 4 *Given the utility function of the representative agent as described by equation (3), then the equilibrium price level is*

$$P_t^c = \frac{\phi}{1-\phi} \frac{q_t^2 \left(\beta + \mu_M^*\right) \left(\beta + \mu_M^* + 2k_q\mu_q\right)}{\left(\beta + \mu_M^*\right) + \left(k_q + 3\sigma_q^2\right) q_t 2k_q\mu_q} \frac{M_t}{Y_t} \tag{61}$$

The stochastic process of the CPI is

$$\frac{dP_t^c}{P_t^c} = \pi_t dt + \sigma_q \sqrt{q_t} \left[1 + \frac{\left(\Delta_q \Psi - \Delta \Psi_q\right)}{\Delta \Psi} q_t\right] dW_{M,t} - \sigma_y \sqrt{x_t} dW_{x,t} \tag{62}$$

where the inflation rate is

$$\begin{aligned} \pi_t &= \mu_M^* - \mu_y + \left(\sigma_y^2 - \eta_y\right) x_t + \frac{\left(\Delta_q \Psi - \Delta \Psi_q\right)}{\Delta \Psi} q_t \left(k_q \left(\mu_q - q_t\right) + \frac{\sigma_q^2 q_t}{2}\right) + \quad (63) \\ &\quad + \frac{\left[2\left(\Delta_{qq}\Psi - \Delta\Psi_{qq}\right) - \Delta_q\Psi + \Delta\Psi_q\right] \sigma_q^2 q_t^{3/2}}{2\Psi^2\Delta} \quad (64) \\ \Delta(q) &= \frac{\phi}{1-\phi} \left[q_t^2 \left(\beta + \mu_M^*\right) \left(\beta + \mu_M^* + 2k_q\mu_q\right)\right] \quad (65) \\ \Psi(q) &= \left(\beta + \mu_M^*\right) + \left(k_q + 3\sigma_q^2\right) q_t 2k_q\mu_q \quad (66) \end{aligned}$$

and $\Delta_q = \frac{\partial \Delta(q)}{\partial q}$, $\Psi_q = \frac{\partial \Psi(q)}{\partial q}$, *and* $\Delta_{qq} = \frac{\partial \Delta(q)}{\partial q}$.

Proof 4 *We start by showing how to get (61). From (59) we have that*

$$\frac{1}{M_t} = e^{-\mu_M^*} q_t^{-2}. \tag{67}$$

Define $G(q) = \frac{1}{q_t^2}$. *Thus by using Ito's Lemma, we get*

$$d\left[\frac{e^{2k_q\mu_q}}{q_t^2}\right] = \frac{2}{q_t}\left(k_q + 3\sigma_q^2\right) e^{2k_q\mu_q t} dt - \frac{2\sigma_q}{q_t\sqrt{q_t}} dW_{M,t}. \tag{68}$$

The expected value is

$$\begin{aligned}
E_t\left[\frac{1}{q_s^2}\right] &= E_t\left[\frac{1}{q_s^2} \mid q_t\right] = \\
&= E_t\left[e^{-2k_q\mu_q s}\left\{\int_t^s d\left[\frac{e^{2k_q\mu_q z}}{q_z^2}\right] + \frac{e^{2k_q\mu_q t}}{q_t^2}\right\} \mid q_t\right] = \\
&= \frac{e^{-2k_q\mu_q(s-t)}}{q_t} + \frac{\left(k_q + 3\sigma_q^2\right)}{q_t k_q \mu_q}\left(1 - e^{-2k_q\mu_q(s-t)}\right).
\end{aligned}$$

Finally, from the first order conditions of the problem of the representative agent, we get

$$\frac{1}{P_t^C Y_t} = \frac{1-\phi}{\phi}\int_t^\infty E_t\left[\frac{1}{q_s^2}\right] e^{-\beta(s-t)-\mu_M^*} ds \tag{69}$$

After solving for the integral, we get equation (62).

To compute the inflation rate, it is enough to apply Ito's lemma to (62) by setting $V\left(M_t, Y_t, q_t\right) = \frac{\Delta(q_t)}{\Psi(q_t)}\frac{M_t}{Y_t}$ *so that*

$$dP_t^c = G_M dM_t + G_Y dY_t + G_q dq_t + \frac{1}{2}\left[G_{YY}\left(dY_t\right)^2 + G_{qq}\left(dq_t\right)^2 + G_{Mq}\left(dM_t\right)\left(dq_t\right)\right]. \tag{70}$$

By taking into account (65), (66), and the definitions for Y_t, q_t and M_t, (54), (58) and (59) respectively, we obtain equation (64). ■

8 The real spot interest rate

In this section we derive the dynamics of the real spot interest rate implied by exogenous the dynamics of the technology process. Since this requires solving the differential equation (55), x_t is Markov and satisfies the necessary technical conditions to apply the Representation Theorem of Feyman-Kac.[4] We can now follow the partial differential equation — PDE — approach to compute the real spot rate.

Theorem 5 *If the technology process follows (55) , then the real spot rate r_t is a*

[4]The drift and volatility terms in (55) must be Lipschitz and bounded on $\Re$.

function $\phi(t, x_t)$ that represents the unique solution to the Kolmogorov PDE

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2 \phi(t,x_t)}{\partial x_t^2} + a(b - x_t)\frac{\partial \phi(t,x_t)}{\partial x_t} + \frac{\partial \phi(t,x_t)}{\partial t} - x_t \phi(t, x_t) = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma + a} \end{cases} \tag{71}$$

where the final condition is the long time spot rate determined in Cox, Ingersoll and Ross (1985). This gives

$$r_t = A(\theta) e^{C(\theta) x_t} \tag{72}$$

where $\theta = T - t$ and

$$\begin{aligned} C(\theta) &= \frac{\sigma_x^2 (2 + a)\left(1 - e^{\gamma\theta}\gamma\right)}{2a\left[\sigma_x^2 - a + e^{\gamma\theta}(a - \sigma_x^2) - (1 + e^{\gamma\theta})\gamma\right]} && (73) \\ A(\theta) &= \frac{1}{\gamma + a}\left\{2^{\chi} ab e^{\nu(\theta)} \gamma^{\zeta}\left[a + \gamma - \sigma_x^2 + e^{\gamma\theta}\left(\gamma - a + \sigma_x^2\right)\right]^{-\zeta}\right\} && (74) \\ \chi &= \frac{a^2 - \gamma^2 - 2\sigma_x^2(a + b + ab) + \sigma^4(1 + b)}{(a - \sigma_x^2)^2 - \gamma^2} && (75) \\ \nu(\theta) &= \frac{b\theta\sigma_x^2(2 + a - \gamma)}{2(a + \gamma - \sigma_x^2)} && (76) \\ \zeta &= \frac{b\sigma_x^2\left(\sigma_x^2 - 2a - 2\right)}{(a - \sigma^2)^2 - \gamma^2} && (77) \\ \gamma &= \sqrt{a^2 + 2\sigma_x^2} && (78) \end{aligned}$$

Proof 5 *If $\phi(t, x_t) = r_t$, like in equation (72), then the Kolmogorov PDE is*

$$\begin{cases} \frac{1}{2}\sigma^2 x_t C^2(\theta) r_t + a(b - x_t) C(\theta) r_t + A'(\theta) e^{C(\theta)x_t} + \\ + C'(\theta) x_t r_t - x_t r_t = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma + a} \end{cases} \tag{79}$$

where $A'(\theta)$ and $C'(\theta)$ represent the derivative with respect to time of functions $A(\theta)$ and $C(\theta)$ equations (73) and (74). Given that the Kolmogorov PDE is verified for all t and x_t, we can divide it into two parts. One part is dependent and the other one is independent from x_t. The task boils down to solving the differential equation

— DE — system

$$\begin{cases} \frac{1}{2}\sigma_x^2 C^2(\theta) - aC(\theta) + C'(\theta) - 1 = 0 & C(0) = 1 \\ A'(\theta) + abA(\theta)C(\theta) = 0 & A(0) = \frac{2ab}{\gamma+a}. \end{cases} \tag{80}$$

The first DE is a Riccati equation whose solution is (73). The second equation, instead, is a regular first order DE with solution given by (74). ■

Lemma 6 *The dynamics of real spot interest rate is*

$$dr_t = a^*(\theta)\left[b^*(\theta) - r_t\right]dt + C(\theta)\, r_t \sigma_x \sqrt{x_t} dW_{x,t} \tag{81}$$

where $(W_{x,t})_t$ *is a unidimensional* $\mathbb{Q}-$*Brownian motion, and with*

$$\begin{aligned} a^*(\theta) &= -\left[a(b - x_t)C(\theta) + C'(\theta)x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right] \\ b^*(\theta) &= \frac{A'(\theta)e^{C(\theta)x_t}}{a^*(\theta)}. \end{aligned}$$

Proof 6 *By applying Ito's Lemma to (72), the dynamics of* $r_t = \phi(t, x_t)$ *is*

$$\begin{aligned} dr_t &= C(\theta) r_t dx_t + \frac{\partial r_t}{\partial t}dt + \frac{1}{2}\sigma_x^2 x_t C^2(\theta) r_t dt & (82) \\ &= r_t\left[a(b - x_t)C(\theta) + C'(\theta)x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right]dt + & (83) \\ &\quad + A'(\theta)e^{C(\theta)x_t}dt + C(\theta) r_t \sigma_x \sqrt{x_t} dW_{t,x} & (84) \end{aligned}$$

Finally, we can write (83) as a mean reverting process in square root with time dependent coefficients like in (81). ■

9 The term structure of real interest rates

Here we show how to compute the price of zero coupon bonds as a function of time, technology and the real spot rate. We follow again a PDE approach because both x_t and r_t are Markov and satisfy the necessary technical conditions to apply the

Representation Theorem of Feyman-Kac. The solution has no closed form, and it is necessary to use numerical methods to understand how it works.

Theorem 7 *If technology and the real spot rate follow (55) and (83), respectively, then the zero coupon bond $B(t,T)$ is a function $\vartheta(t,x_t,r_t)$ that represents the unique solution to the Kolmogorov PDE*

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial x_t^2} + a\left[b-x_t\right]\frac{\partial\vartheta(t,x_t,r_t)}{\partial x_t} + \frac{1}{2}C^2(\theta)\,C^{*2}(\theta)\,r_t^2\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial r_t^2} + \\ +a^*(\theta)\left[b^*(\theta)-r_t\right]\frac{\partial\vartheta(t,x_t,r_t)}{\partial r_t} + r_t C(\theta)\,\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial r_t \partial x_t} + \frac{\partial\vartheta(t,x_t,r_t)}{\partial t} + \\ -r_t\vartheta(t,x_t,r_t) = 0 \\ \vartheta(T,x_t,r_t) = 1 \end{cases} \tag{85}$$

Proof 7 *Let us assume $\vartheta(t,x_t,r_t) = B(t,T)$ with*

$$B(t,T) = A^*(\theta)\,e^{-C^*(\theta)r_t}. \tag{86}$$

The Kolmogorov PDE becomes

$$\begin{cases} -\frac{1}{2}\sigma_x^2 x_t C^2(\theta)\,r_t C^{*2}(\theta)\,B(t,T)\left[1-C^{*2}(\theta)\,r_t\right] + \\ -a\left[b-x_t\right]C^*(\theta)\,B(t,T)\,r_t C(\theta) + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\,r_t^2 C^{*2}(\theta)\,B(t,T) + \\ -a^*(\theta)\left[b^*(\theta)-r_t\right]C^*(\theta)\,B(t,T) + \sigma_x^2 x_t C^2(\theta)\,r_t^2 C^{*2}(\theta)\,B(t,T) + \\ +A^{*'}(\theta)\,e^{C^*(\theta)r_t} - C^{*'}(\theta)\,r_t B(t,T) - r_t B(t,T) = 0 \\ \vartheta(T,x_t,r_t) = 1. \end{cases} \tag{87}$$

Since the PDE (87) is verified for all t, x_t and r_t, we can divide it into two equations, one dependent and one independent from x_t and r_t. Now the problem is to solve the DE system

$$\begin{cases} \frac{1}{2}\sigma^{*2}(\theta)\,C^{*2}(\theta) - \Psi(t)\,C^*(\theta) - C^{*'}(\theta) - 1 = 0 & C^*(0) = 0 \\ A^{*'}(\theta) - a^*(\theta)\,b^*(\theta)\,A^*(\theta)\,C^*(\theta) = 0 & A^*(0) = 1 \end{cases} \tag{88}$$

where $\sigma^*(t)$ and $\Psi(t)$ are

$$\sigma^*(\theta) = 2\sigma_x C(\theta)\sqrt{r_t x_t} \tag{89}$$

$$\Psi(\theta) = \frac{1}{2}\sigma_x^2 x_t C^2(\theta) + abC(\theta) - aC(\theta)x_t - a^*(\theta). \tag{90}$$

The first DE is a Riccati equation, and the second is a first order DE. Both equations have time-varying coefficient. We can then find the solution only by using numerical methods. ■

Lemma 8 *The term structure of real interest rates is*

$$R(t,T) = -\frac{1}{\theta}\left[\ln A^*(\theta) - r_t C^*(\theta)\right]. \tag{91}$$

where $A^*(\theta)$ and $C^*(\theta)$ are the solutions of system (88).

Proof 8 *The term structure of real interest rates can be derived from the relation between the price of zero coupon bond and the continuous real interest rate*

$$B(t,T) = e^{-\theta R(t,T)}. \tag{92}$$

■

10 The term structure of nominal interest rates

In this section we derive the equilibrium nominal spot interest rate, the analytical expression for the nominal zero coupon bond, and for the nominal term structure of interest rates.

Lemma 9 *The nominal spot interest rate is*

$$i_t = A(\theta)e^{C(\theta)x_t}\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2(\beta+\mu_M^*)(\beta+\mu_M^*+2k_q\mu_q)}{(\beta+\mu_M^*)+(k_q+3\sigma_q^2)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{93}$$

where $A(\theta)$, $C(\theta)$ are (73) and (74), respectively.

Proof 9 *Multiply (72) by (61).* ■

Lemma 10 *The nominal zero coupon bond is*

$$N\left(t,T\right)=B\left(t,T\right)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{94}$$

where $B\left(t,T\right)$ is the solution of (87) .

Proof 10 *Multiply the solution of (85) by (61).* ■

Lemma 11 *The nominal term structure of interest rate is*

$$I\left(t,T\right)=R\left(t,T\right)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{95}$$

where $R\left(t,T\right)$ is (91)

Proof 11 *Multiply (91) by (61).* ■

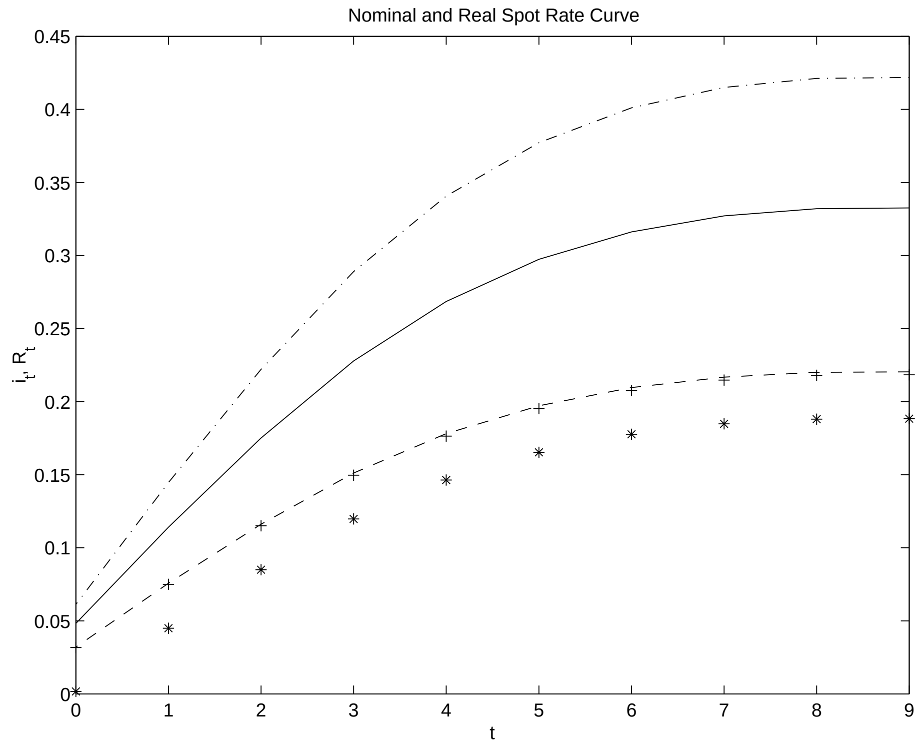
From the expressions for (93), (94) and (95), we observe that the values of the rates for the nominal term structure are higher than those for the real variables if (61) is higher than one. This is linked to ϕ_1, which determines μ_M^*), through the condition

$$\mu_M^*<\mu_M^{*a} \text{ and } \mu_M^*>\mu_M^{*b} \tag{96}$$

where μ_M^{*a}, μ_M^{*b} are

$$\begin{aligned}\mu_M^{*a,b} &= \frac{-\Gamma\pm\sqrt{\Gamma^2+4M_t\phi q_t^2\Lambda}}{2M_t\phi q_t^2} && (97)\\ \Gamma &= 2M_t\phi q_t^2\left(\beta+k_q\mu_q\right)-Y_t\left(1-\phi\right) && (98)\\ \Lambda &= -Y_t\left(1-\phi\right)\beta-2Y_tk_q\mu_qq_t\left(k_q+3\sigma_q^2\right)\left(1-\phi\right) && (99)\\ &\qquad +M_t\phi q_t^2\beta\left(\beta+2k_q\mu_q\right)\end{aligned}$$

Figure 1: The nominal and real spot curve



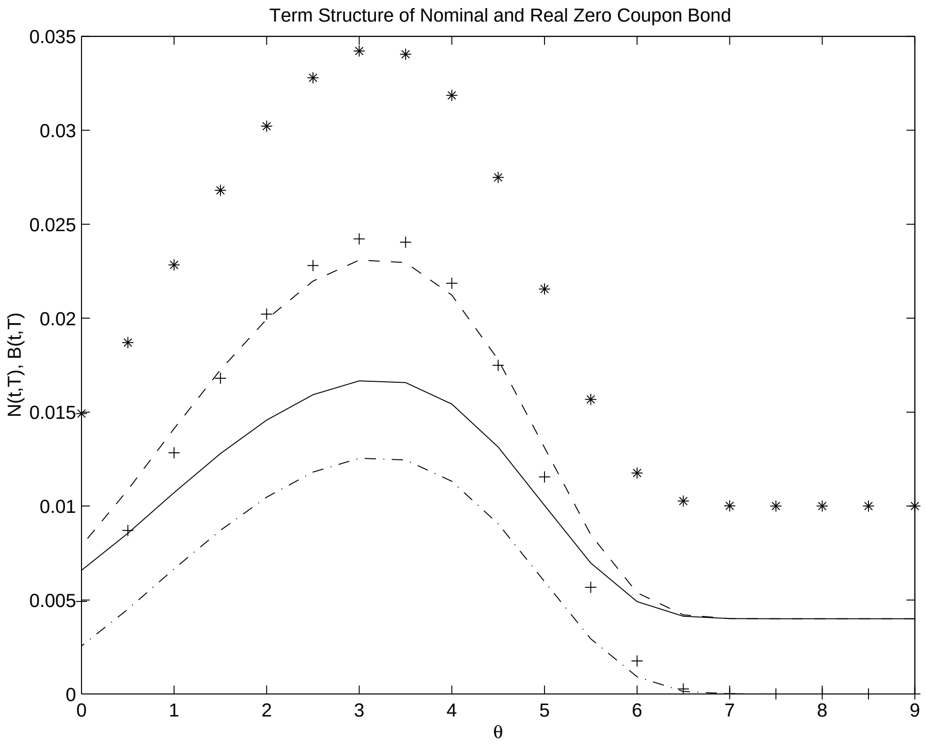
Legend: The line (−.) is for $\mu_M^* = 0.98$, the dark line is for $\mu_M^* = 0.55$, the dashed (−−) is for $\mu_M^* = 0.01$ and the crossed curve (++) is for $\mu_M^* = 0.0001$.

11 Simulation results

We now calibrate the model and run numerical simulations to get a better understanding of the relations involved. The intertemporal substitution coefficient β has been set equal to 0.998. The values for the other parameters are from Balduzzi (1998), who proposes a model with similar stochastic processes. In particular, we have $\kappa_q = 0.3$, $\mu_q = 0.1$, $\mu_Y = 0.2$, $\eta_Y = 0.4$, $\sigma_q = 0.1$, $\phi = 0.5$, $\mu_M = 0.2$. For the other parameters we have chosen values that are consistent from an economic point of view, that is $a = 0.45$, $b = 0.03$, $\sigma_x = 1.35$. The solution of the model on the stochastic processes for x, q, M and Y. We initialize these processes at $x_0 = 0.1$, $q_0 = 0.1$, $M_0 = 0.1$ and $Y_0 = 1$.

Figure 1 plots the nominal and real spot rate curve for different values of μ_M^*.

Figure 2: The nominal and real term structure of zero coupon bonds



Legend: The line (−.) is for $\mu_M^* = 0.98$, the dark line is for $\mu_M^* = 0.55$, the dashed (−−) is for $\mu_M^* = 0.01$ and the crossed curve (++) is for $\mu_M^* = 0.0001$.

The curve of real rates lays below the nominal curve.[5] The parameters governing the real curve are kept constant. However, we impose the constraint in (96). Figure 1 shows that the higher the reaction of the tax rate to real debt, the lower μ_M^*, and the lower the position of the nominal curve in the plan. This implies that the average nominal rates are consistent with the notion of equilibrium outlined earlier. For a very elastic tax rate (i.e. a very low μ_M^*), the curve of nominal spot rates does not change as a function of fiscal parameters.

In order to simulate a closed form solution for the term structure of zero-coupon bonds, we assume that the coefficients of the equations in (88) are constant. We

[5]The case where the nominal curve is above the real curve can be thought as the result of a very tight fiscal policy that produces a deflationary equilibrium. Such deflationary equilibria are not to be ruled out within the framework of the fiscal theory of price level determination. They can be interpreted as arising from the contraction of aggregate demand because of a high of taxes to real debt.

are aware of the limitation of this choice. However, this is only meant to make the discussion of the results more intuitive.

Figure 2 shows that the position of the term structure depends on the value assumed for μ_M^* and, consequently, on the value of the tax elasticity ϕ_1. If the tax elasticity falls and μ_M^* rises, the curve of the nominal zero coupon bond shifts downward except for very low values of μ_M^*, when the curve loses sensitiveness to this parameter. This is consistent with the results form the fiscal theory of the price level. As the tax rate falls, also the price for newly issued debt drops because the reduced backing of taxes generates inflationary risk in the future, thus causing a reduction of the nominal value of debt today.

12 Concluding remarks

In this paper we study a simple intertemporal model for the determination of the nominal and real term structure where the interaction between fiscal and monetary plays a key role. In particular, we investigate the relation between the term structure of interest rate and the fiscal theory of price level determination. In so doing, we move beyond the standard finance models where monetary and technological factors are the sole determinants of the term structure of interest rates.

A number of interesting avenues of future work can be considered. The model presented in this paper should be taken to the data to study how inflation risk premia are affected by fiscal determinants. An ongoing work considers the pricing of interest-rate futures in our model. This is likely to shadow more light on the role of monetary policy expectations when also fiscal policy matters.

A Appendix

Proposition 12 *In the continuous time limit equilibrium, the real interest rate has the form*

$$\begin{aligned} r_t &= \beta - \frac{Y_t u_{cc}}{u_c}\frac{1}{dt}E_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{1}{2}\frac{Y_t^2 u_{cc}}{u_c} var_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{m_t u_{cm}}{u_c}\frac{1}{dt}E_t\left\{\frac{dm_t}{m_t}\right\} \quad (100) \\ & -\frac{1}{2}\frac{m_t^2 u_{cmm}}{u_c} var_t\left\{\frac{dm_t}{m_t}\right\} - \frac{Y_t m_t u_{ccm}}{u_c} cov_t\left\{\frac{dY_t}{Y_t}, \frac{dm_t}{m_t}\right\}. \quad (101) \end{aligned}$$

Proof 12 *A Taylor expansion around the equilibrium of first order condition (32) yields*

$$\begin{aligned} u_c(Y_t, m_t)(1 + r_t\Delta t) = E_t\{(1 + r_t\Delta t)[u_c(Y_t, m_t) + u_{cc}(Y_t, m_t)\Delta Y_t + \\ + u_{cm}(Y_t, m_t)\Delta m_t + \frac{u_{ccc}(Y_t, m_t)}{2}(\Delta Y_t)^2 \\ + \frac{u_{ccm}(Y_t, m_t)}{2}\Delta Y_t \Delta m_t + \frac{u_{cmm}(Y_t, m_t)}{2}(\Delta m_t)^2\Big]\Big\} + o(\Delta t)^{3/2}. \quad (102) \end{aligned}$$

After collecting terms and re-arranging

$$\begin{aligned} r_t = \beta - \frac{1}{\Delta t}E_t\Bigg\{\frac{u_{cc}Y_t}{u_c}\left(\frac{\Delta Y_t}{Y_t}\right) + \frac{u_{cm}}{u_c}m_t\left(\frac{\Delta m_t}{m_t}\right) + \frac{u_{ccc}Y_t^2}{2u_c}\left(\frac{\Delta Y_t}{Y_t}\right)^2 + \\ + \frac{u_{ccm}Y_t m_t}{2u_c}\left(\frac{\Delta Y_t}{Y_t}\right)\left(\frac{\Delta m_t}{m_t}\right) + \frac{u_{cmm}m_t^2}{2u_c}\left(\frac{\Delta m_t}{m_t}\right)^2\Bigg\} + o(\Delta t)^{3/2}. \quad (103) \end{aligned}$$

Thus, take the limit of (103) and apply Ito's multiplication rule, we get equation (101), after having recalled that

$$\begin{aligned} \frac{1}{dt}E_t\left\{\left(\frac{\Delta Y_t}{Y_t}\right)^2\right\} &= var_t\left(\frac{\Delta Y_t}{Y_t}\right) \\ \frac{1}{dt}E_t\left\{\left(\frac{\Delta m_t}{m_t}\right)^2\right\} &= var_t\left(\frac{\Delta m_t}{m_t}\right). \end{aligned}$$

■

Corollary 13 *The real spot interest rate determined in equation (72) is also an* *equilibrium rate, for utility function parameter values ϕ such that this equation and (101) are equated. Consequently, this be also true for the nominal spot interest rate.*

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